各問のベクトル場 $\vec{\mathbf{A}}$ について $div\,\vec{\mathbf{A}}$ と $rot\,\vec{\mathbf{A}}$ を計算せよ。

(1)

$$\vec{\mathbf{A}}(x,y,z) = z\,\vec{\mathbf{i}} + y\,\vec{\mathbf{j}} - x\,\vec{\mathbf{k}}$$

(2)

$$\vec{\mathbf{A}}(x,y,z) = xy\,\vec{\mathbf{i}} + yz\,\vec{\mathbf{j}} - xz\,\vec{\mathbf{k}}$$

(3)

$$\vec{\mathbf{A}}(x,y,z) = 3yz\,\vec{\mathbf{i}} - xz\,\vec{\mathbf{j}} + 2xy\,\vec{\mathbf{k}}$$

(4)

$$\vec{\mathbf{A}}(x,y,z) = x\,\vec{\mathbf{i}} + y\,\vec{\mathbf{j}} - z\,\vec{\mathbf{k}}$$

(5)

$$\vec{\mathbf{A}}(x,y,z) = 2xz\,\vec{\mathbf{i}} + xy\,\vec{\mathbf{j}} - 3xz\,\vec{\mathbf{k}}$$

(6)

$$\vec{\mathbf{A}} = xy\,\vec{\mathbf{i}} + 3yz\,\vec{\mathbf{j}} + 2xz\,\vec{\mathbf{k}}$$

(7)

$$\vec{\mathbf{A}} = 3x^2\,\vec{\mathbf{i}} + y^2\,\vec{\mathbf{j}} + \left(xz + 4z^2\right)\vec{\mathbf{k}}$$

(8)

$$\vec{\mathbf{A}} = (x + 2y + z)\,\vec{\mathbf{i}} + (x + y + z)\,\vec{\mathbf{j}} + \left(2x + y + 3z\right)\vec{\mathbf{k}}$$

(9)

$$\vec{\mathbf{A}} = x^2yz\,\vec{\mathbf{i}} + xy^2z\,\vec{\mathbf{j}} + xyz^2\,\vec{\mathbf{k}}$$

(10)

$$\vec{\mathbf{A}} = \left(2xy - y^2\right)\vec{\mathbf{i}} + \left(yz + z^2\right)\vec{\mathbf{j}} + 3xz\,\vec{\mathbf{k}}$$

(1)

$$\vec{\mathbf{A}}(x,y,z)=z\,\vec{\mathbf{i}}+y\,\vec{\mathbf{j}}-x\vec{\mathbf{k}}$$

$$\mathrm{div}\,\vec{\mathbf{A}}=\frac{\partial}{\partial x}(z)\,\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}+\frac{\partial}{\partial y}(y)\,\vec{\mathbf{j}}\cdot\vec{\mathbf{j}}+\frac{\partial}{\partial z}(-x)\vec{\mathbf{k}}\cdot\vec{\mathbf{k}}=1$$

$$\mathrm{rot}\,\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z & y & -x\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & -x\end{vmatrix}\vec{\mathbf{i}}+\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ -x & z\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ z & y\end{vmatrix}\vec{\mathbf{k}}$$

$$=\left\{\frac{\partial}{\partial y}(-x)-\frac{\partial}{\partial z}(y)\right\}\vec{\mathbf{i}}+\left\{\frac{\partial}{\partial z}(z)-\frac{\partial}{\partial x}(-x)\right\}\vec{\mathbf{j}}+\left\{\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(z)\right\}\vec{\mathbf{k}}$$

$$=2\,\vec{\mathbf{j}}$$

(2)

$$\vec{\mathbf{A}}(x,y,z)=xy\,\vec{\mathbf{i}}+yz\,\vec{\mathbf{j}}-xz\vec{\mathbf{k}}$$

$$\mathrm{div}\,\vec{\mathbf{A}}=\frac{\partial}{\partial x}(xy)\,\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}+\frac{\partial}{\partial y}(yz)\,\vec{\mathbf{j}}\cdot\vec{\mathbf{j}}+\frac{\partial}{\partial z}(-xz)\vec{\mathbf{k}}\cdot\vec{\mathbf{k}}=y+z-x=-x+y+z$$

$$\mathrm{rot}\,\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & yz & -xz\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & -xz\end{vmatrix}\vec{\mathbf{i}}+\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ -xz & xy\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ xy & yz\end{vmatrix}\vec{\mathbf{k}}$$

$$=\left\{\frac{\partial}{\partial y}(-xz)-\frac{\partial}{\partial z}(yz)\right\}\vec{\mathbf{i}}+\left\{\frac{\partial}{\partial z}(xy)-\frac{\partial}{\partial x}(-xz)\right\}\vec{\mathbf{j}}+\left\{\frac{\partial}{\partial x}(yz)-\frac{\partial}{\partial y}(xy)\right\}\vec{\mathbf{k}}$$

$$=-y\vec{\mathbf{i}}+z\,\vec{\mathbf{j}}-x\vec{\mathbf{k}}$$

(3)

$$\vec{\mathbf{A}}(x,y,z)=3yz\,\vec{\mathbf{i}}-xz\,\vec{\mathbf{j}}+2xy\vec{\mathbf{k}}$$

$$\mathrm{div}\,\vec{\mathbf{A}}=\frac{\partial}{\partial x}(3yz)\,\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}+\frac{\partial}{\partial y}(-xz)\,\vec{\mathbf{j}}\cdot\vec{\mathbf{j}}+\frac{\partial}{\partial z}(2xy)\vec{\mathbf{k}}\cdot\vec{\mathbf{k}}=0$$

$$\mathrm{rot}\,\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3yz & -xz & 2xy\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -xz & 2xy\end{vmatrix}\vec{\mathbf{i}}+\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ 2xy & 3yz\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ 3yz & -xz\end{vmatrix}\vec{\mathbf{k}}$$

$$=\left\{\frac{\partial}{\partial y}(2xy)-\frac{\partial}{\partial z}(-xz)\right\}\vec{\mathbf{i}}+\left\{\frac{\partial}{\partial z}(3yz)-\frac{\partial}{\partial x}(2xy)\right\}\vec{\mathbf{j}}+\left\{\frac{\partial}{\partial x}(-xz)-\frac{\partial}{\partial y}(3yz)\right\}\vec{\mathbf{k}}$$

$$=3x\vec{\mathbf{i}}+y\vec{\mathbf{j}}-4z\vec{\mathbf{k}}$$

(4)

$$\vec{\mathbf{A}}(x,y,z)=x\,\vec{\mathbf{i}}+y\,\vec{\mathbf{j}}-z\,\vec{\mathbf{k}}$$

$$\mathrm{div}\,\vec{\mathbf{A}}=\frac{\partial}{\partial x}(x)\,\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}+\frac{\partial}{\partial y}(y)\,\vec{\mathbf{j}}\cdot\vec{\mathbf{j}}+\frac{\partial}{\partial z}(-z)\vec{\mathbf{k}}\cdot\vec{\mathbf{k}}=1+1-1=1$$

$$\mathrm{rot}\,\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & -z\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & -z\end{vmatrix}\vec{\mathbf{i}}+\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ -z & x\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ x & y\end{vmatrix}\vec{\mathbf{k}}$$

$$=\left\{\frac{\partial}{\partial y}(-z)-\frac{\partial}{\partial z}(y)\right\}\vec{\mathbf{i}}+\left\{\frac{\partial}{\partial z}(x)-\frac{\partial}{\partial x}(-z)\right\}\vec{\mathbf{j}}+\left\{\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(x)\right\}\vec{\mathbf{k}}$$

$$=0$$

(5)

$$\vec{\mathbf{A}}(x,y,z)=2xz\,\vec{\mathbf{i}}+xy\,\vec{\mathbf{j}}-3xz\,\vec{\mathbf{k}}$$

$$\mathrm{div}\,\vec{\mathbf{A}}=\frac{\partial}{\partial x}(2xz)\,\vec{\mathbf{i}}\cdot\vec{\mathbf{i}}+\frac{\partial}{\partial y}(xy)\,\vec{\mathbf{j}}\cdot\vec{\mathbf{j}}+\frac{\partial}{\partial z}(-3xz)\vec{\mathbf{k}}\cdot\vec{\mathbf{k}}=2z+x-3x=2z-2x$$

$$\mathrm{rot}\,\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xz & xy & -3xz\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & -3xz\end{vmatrix}\vec{\mathbf{i}}+\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ -3xz & 2xz\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ 2xz & xy\end{vmatrix}\vec{\mathbf{k}}$$

$$=\left\{\frac{\partial}{\partial y}(-3xz)-\frac{\partial}{\partial z}(xy)\right\}\vec{\mathbf{i}}+\left\{\frac{\partial}{\partial z}(2xz)-\frac{\partial}{\partial x}(-3xz)\right\}\vec{\mathbf{j}}+\left\{\frac{\partial}{\partial x}(xy)-\frac{\partial}{\partial y}(2xz)\right\}\vec{\mathbf{k}}$$

$$=(2x+3z)\vec{\mathbf{j}}+y\,\vec{\mathbf{k}}$$

(6)

$$\vec{\mathbf{A}}=xy\;\vec{\mathbf{i}}+3yz\,\vec{\mathbf{j}}+2xz\,\vec{\mathbf{k}}$$

$$\mathrm{div}\,\vec{\mathbf{A}}=\frac{\partial}{\partial x}(xy)+\frac{\partial}{\partial y}(3yz)+\frac{\partial}{\partial z}(2xz)=y+3z+2x$$

$$\mathrm{rot}\,\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 3yz & 2xz\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3yz & 2xz\end{vmatrix}\vec{\mathbf{i}}+\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ 2xz & xy\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ xy & 3yz\end{vmatrix}\vec{\mathbf{k}}$$

$$=\left\{\frac{\partial}{\partial y}(2xz)-\frac{\partial}{\partial z}(3yz)\right\}\vec{\mathbf{i}}+\left\{\frac{\partial}{\partial z}(xy)-\frac{\partial}{\partial x}(2xz)\right\}\vec{\mathbf{j}}+\left\{\frac{\partial}{\partial x}(3yz)-\frac{\partial}{\partial y}(xy)\right\}\vec{\mathbf{k}}$$

$$=(0-3y)\,\vec{\mathbf{i}}+(0-2z)\,\vec{\mathbf{j}}+(0-x)\,\vec{\mathbf{k}}=-3y\vec{\mathbf{i}}-2z\vec{\mathbf{j}}-x\vec{\mathbf{k}}$$

(7)

$$\vec{\mathbf{A}} = 3x^2\,\vec{\mathbf{i}} + y^2\,\vec{\mathbf{j}} + \left(xz + 4z^2\right)\vec{\mathbf{k}}$$

$$\mathrm{div}\ \vec{\mathbf{A}} = \frac{\partial}{\partial x}\left(3x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(xz + 4z^2\right) = 6x + 2y + x + 8z = 7x + 2y + 8z$$

$$\mathrm{rot}\ \vec{\mathbf{A}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 & y^2 & (xz + 4z^2) \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & (xz + 4z^2) \end{vmatrix} \vec{\mathbf{i}} + \begin{vmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \\ (xz + 4z^2) & 3x^2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 3x^2 & y^2 \end{vmatrix} \vec{\mathbf{k}}$$

$$= \left\{\frac{\partial}{\partial y}(xz + 4z^2) - \frac{\partial}{\partial z}(y^2)\right\}\vec{\mathbf{i}} + \left\{\frac{\partial}{\partial z}(3x^2) - \frac{\partial}{\partial x}(xz + 4z^2)\right\}\vec{\mathbf{j}} + \left\{\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(3x^2)\right\}\vec{\mathbf{k}}$$

$$= -z\vec{\mathbf{j}}$$

(8)

$$\vec{\mathbf{A}} = (x + 2y + z)\,\vec{\mathbf{i}} + (x + y + z)\,\vec{\mathbf{j}} + \left(2x + y + 3z\right)\vec{\mathbf{k}}$$

$$\mathrm{div}\ \vec{\mathbf{A}} = \frac{\partial}{\partial x}\left(x + 2y + z\right) + \frac{\partial}{\partial y}\left(x + y + z\right) + \frac{\partial}{\partial z}\left(2x + y + 3z\right) = 1 + 1 + 3 = 5$$

$$\mathrm{rot}\ \vec{\mathbf{A}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + z & x + y + z & 2x + y + 3z \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + y + z & 2x + y + 3z \end{vmatrix} \vec{\mathbf{i}} + \begin{vmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \\ 2x + y + 3z & x + 2y + z \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x + 2y + z & x + y + z \end{vmatrix} \vec{\mathbf{k}}$$

$$= \left\{\frac{\partial}{\partial y}(2x + y + 3z) - \frac{\partial}{\partial z}(x + y + z)\right\}\vec{\mathbf{i}} + \left\{\frac{\partial}{\partial z}(x + 2y + z) - \frac{\partial}{\partial x}(2x + y + 3z)\right\}\vec{\mathbf{j}}$$

$$+ \left\{\frac{\partial}{\partial x}(x + y + z) - \frac{\partial}{\partial y}(x + 2y + z)\right\}\vec{\mathbf{k}}$$

$$= (1 - 1)\,\vec{\mathbf{i}} + (1 - 2)\,\vec{\mathbf{j}} + (1 - 2)\,\vec{\mathbf{k}} = -\vec{\mathbf{j}} - \vec{\mathbf{k}}$$

(9)

$$\vec{\mathbf{A}} = x^2yz\,\vec{\mathbf{i}} + xy^2z\,\vec{\mathbf{j}} + xyz^2\,\vec{\mathbf{k}}$$

$$\text{div}\ \vec{\mathbf{A}} = \frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(xy^2z\right) + \frac{\partial}{\partial z}\left(xyz^2\right) = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\text{rot}\ \vec{\mathbf{A}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & xyz^2 \end{vmatrix}\vec{\mathbf{i}} + \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ xyz^2 & x^2yz \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2yz & xy^2z \end{vmatrix}\vec{\mathbf{k}}$$

$$= \left\{\frac{\partial}{\partial y}(xyz^2) - \frac{\partial}{\partial z}(xy^2z)\right\}\vec{\mathbf{i}} + \left\{\frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}(xyz^2)\right\}\vec{\mathbf{j}} + \left\{\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(x^2yz)\right\}\vec{\mathbf{k}} \ (10)$$

$$= (xz^2 - xy^2)\,\vec{\mathbf{i}} + (x^2y - yz^2)\,\vec{\mathbf{j}} + (y^2z - x^2z)\vec{\mathbf{k}}$$

$$= x(z^2 - y^2)\,\vec{\mathbf{i}} + y(x^2 - z^2)\,\vec{\mathbf{j}} + z(y^2 - x^2)\vec{\mathbf{k}}$$

$$\vec{\mathbf{A}} = \left(2xy - y^2\right)\vec{\mathbf{i}} + \left(yz + z^2\right)\vec{\mathbf{j}} + 3xz\vec{\mathbf{k}}$$

(10)

$$\vec{\mathbf{A}} = \left(2xy - y^2\right)\vec{\mathbf{i}} + \left(yz + z^2\right)\vec{\mathbf{j}} + 3xz\vec{\mathbf{k}}$$

$$\text{div}\ \vec{\mathbf{A}} = \frac{\partial}{\partial x}\left(2xy - y^2\right) + \frac{\partial}{\partial y}\left(yz + z^2\right) + \frac{\partial}{\partial z}\left(3xz\right) = 2y + z + 3x$$

$$\text{rot}\ \vec{\mathbf{A}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - y^2 & yz + z^2 & 3xz \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + z^2 & 3xz \end{vmatrix}\vec{\mathbf{i}} + \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ 3xz & 2xy - y^2 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy - y^2 & yz + z^2 \end{vmatrix}\vec{\mathbf{k}}$$

$$= \left\{\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(yz + z^2)\right\}\vec{\mathbf{i}} + \left\{\frac{\partial}{\partial z}(2xy - y^2) - \frac{\partial}{\partial x}(3xz)\right\}\vec{\mathbf{j}}$$

$$+ \left\{\frac{\partial}{\partial x}(yz + z^2) - \frac{\partial}{\partial y}(2xy - y^2)\right\}\vec{\mathbf{k}}$$

$$= (0 - y - 2z)\,\vec{\mathbf{i}} + (0 - 3z)\,\vec{\mathbf{j}} + (0 - 2x + 2y)\vec{\mathbf{k}} = -(y + 2z)\vec{\mathbf{i}} - 3z\vec{\mathbf{j}} + 2(y - x)\vec{\mathbf{k}}$$